9/3 Linear eqⁿ

(1)

It is in the form of

$\frac{dy}{dx} + Py = Q$ as linear eqⁿ.

then $IF = e^{\int P dx}$

So the solution of this eqⁿ (formulla)

$$\boxed{y.IF = \int Q.IF.\ dx + C}$$

or $\frac{dx}{dy} + Px = Q$ then $IF = e^{\int P dy}$

So solution is

$$\boxed{x.IF = \int Q.IF\ dy + C}$$

Q $x \log x \frac{dy}{dx} + y = \log x^2$.

$$\frac{dy}{dx} + \frac{1}{x \log x} y = \frac{\log x^2}{x \log x}$$

compare this eqⁿ $\frac{dy}{dx} + Py = Q$

$$P = \frac{1}{x \log x}, \quad Q = \frac{\log x^2}{x \log x}$$

$$IF = e^{\int P dx} = e^{\int \frac{1}{x \log x} dx} = e^{\log(\log x)} = \underline{\log x}$$

So solution is

$$y.\log x = \int \cancel{\log x} \frac{\log x^2}{x \cancel{\log x}} dx + C$$

let $\log x^2 = t$

$\frac{1}{x^2} 2x\ dx = dt$

$$y \log x = \int \frac{t}{2}\, dt \qquad \frac{dx}{x} = \frac{dt}{2}$$

$$y \log x = \frac{t^2}{4} + c$$

$$\boxed{y \times \log x = \frac{(\log x)^2}{4} + c}$$

Q $\sqrt{1-y^2}\, dx = (\sin^{-1} y - x)\, dy$

$$\sqrt{1-y^2} = (\sin^{-1} y - x)\frac{dy}{dx}$$

or $\sqrt{1-y^2}\,\frac{dx}{dy} = \sin^{-1} y - x.$

$$\sqrt{1-y^2}\,\frac{dx}{dy} + x = \sin^{-1} y$$

$$\frac{dx}{dy} + \frac{1}{\sqrt{1-y^2}}\, x = \frac{\sin^{-1} y}{\sqrt{1-y^2}} \quad \text{--- (1)}$$

$$P = \frac{1}{\sqrt{1-y^2}}, \quad Q = \frac{\sin^{-1} y}{\sqrt{1-y^2}}$$

$$I.F = e^{\int \frac{1}{\sqrt{1-y^2}} dy} = e^{\sin^{-1} y}$$

thus solution $x . I.F = \int Q . I.F\, dy + c$

$$x . e^{\sin^{-1} y} = \int \frac{\sin^{-1} y}{\sqrt{1-y^2}}\, e^{\sin^{-1} y}\, dy + c$$

$$= \int t\, e^{t}\, dt + c \qquad \sin^{-1} y = t, \quad \frac{1}{\sqrt{1-y^2}}\, dy = dt$$

② $e^{\sin^{-1}y} = te^t - e^t + c$

$$x e^{\sin^{-1}y} = \sin^{-1}y\, e^{\sin^{-1}y} - e^{\sin^{-1}y} + C$$

$$\boxed{x = \sin^{-1}y - 1 + C/e^{\sin^{-1}y}}$$

Q $(1+y^2)\,dx + (x - e^{\tan^{-1}y})\,dy = 0$

or $(1+y^2)\dfrac{dx}{dy} + x - e^{\tan^{-1}y} = 0$

$$\frac{dx}{dy} + \frac{1}{1+y^2}x = \frac{e^{\tan^{-1}y}}{1+y^2}$$

$$P = \frac{1}{1+y^2}, \quad Q = \frac{e^{\tan^{-1}y}}{1+y^2}$$

$$If = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$$

Thus the solution

$$x.If = \int Q.If\, dy + c$$

$$x e^{\tan^{-1}y} = \int \frac{e^{\tan^{-1}y}}{1+y^2} \cdot e^{\tan^{-1}y}\, dy + c$$

$$x e^{\tan^{-1}y} = \int \frac{1}{1+y^2}dy + c$$

$$\boxed{x e^{\tan^{-1}y} = \tan^{-1}y + c}$$

Q $$\frac{dy}{dx} = -\frac{x + y\cos x}{1+\sin x}$$

or $$(1+\sin x)\frac{dy}{dx} = -(x + y\cos x)$$

$$(1+\sin x)\frac{dy}{dx} + y\cos x = -x$$

$$\frac{dy}{dx} + \frac{\cos x}{1+\sin x}y = -\frac{x}{1+\sin x}$$

$$P = \frac{\cos x}{1+\sin x}, \quad Q = -\frac{x}{1+\sin x}$$

$$If = e^{\int P dx} = e^{\int \frac{\cos x}{1+\sin x}dx} = e^{\log(1+\sin x)}$$

$$= (1+\sin x)$$

then solution

$$y.(1+\sin x) = \int -\frac{x}{\cancel{(1+\sin x)}}\cancel{(1+\sin x)}\,dx + C$$

$$\boxed{y(1+\sin x) = -\frac{x^2}{2} + C}$$

Q $\cos^2 x \frac{dy}{dx} + y = \tan x$

Q $\frac{dy}{dx} - \frac{2y}{x+1} = (x+1)^3$

Q $(x+2y^3)\frac{dy}{dx} = y$ or $y\frac{dx}{dy} = x+2y^3$

or $\frac{dx}{dy} - \frac{x}{y} = 2y^2$

Berno...

(5)

(5) It a linear eqn is in form of

$$\frac{dy}{dx} + PY = QY^n$$ then it is Bernoulli eqn.

so firstly we convert it into linear form then we can solve it.

Q $$\frac{dy}{dx} = y\tan x - y^2 \sec x \quad \text{--- (1)}$$

$$\frac{dy}{dx} - y\tan x = -y^2 \sec x$$

divided by $-y^2$

$$-\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{y}\tan x = \sec x \quad \text{--- (2)}$$

let $\frac{1}{y} = z$ then $-\frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}$

So eqn (2)

$$\frac{dz}{dx} + z\tan x = \sec x \quad \text{--- (3)}$$

$P = \tan x, \quad Q = \sec x$

$$IF = e^{\int \tan x\, dx} = e^{-\log\cos x} = \sec x$$

Solution

$$z \cdot \sec x = \int \sec x \cdot \sec x\, dx + c$$

$$\boxed{z \cdot \sec x = \tan x + c}$$

$\frac{1}{y}\sec x = \tan x + c$

$$\boxed{\frac{1}{y} = \sin x + c\cos x}$$

Q $(x^3y^2 + xy)\,dx = dy$

or $\frac{dy}{dx} - xy = x^3y^2$

or $\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y}x = x^3$ —— ①

let $-\frac{1}{y} = z \Rightarrow -\frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}$

so eqⁿ ①

$-\frac{dz}{dx} - \cancel{z}x = x^3$

or $\frac{dz}{dx} + zx = -x^3$

$P = x$, $\quad IF = e^{\int P\,dx} = e^{\int x\,dx} = e^{x^2/2}$

Now $z \cdot e^{x^2/2} = -\int x^3 e^{x^2/2}\,dx$ $\qquad$ let $\frac{x^2}{2} = t$

$= -\int 2t\,e^t\,dt$ $\qquad$ $x^2 = 2t$, $\cancel{2}x\,dx = \cancel{2}\,dt$

$z\,e^{x^2/2} = -2[te^t - e^t] + c$

$\frac{1}{y}e^{x^2/2} = -2\left[\frac{x^2}{2} - 1\right]e^{x^2/2} + c$

$$\boxed{\frac{1}{y} = (2 - x^2) + c\,e^{-x^2/2}}$$

Q $\sec^2 y \frac{dy}{dx} + x\tan y = x^3$ (7)

Sol Let $\tan y = z \Rightarrow \sec^2 y \frac{dy}{dx} = \frac{dz}{dx}$

Soln $\frac{dz}{dx} + xz = x^3$

$IF = e^{\int x\,dx} = e^{x^2/2}$

$z \cdot e^{x^2/2} = \int x^3 e^{x^2/2} dx + c$

$= \int 2t e^t dt + c$ $\quad \frac{x^2}{2} = t$

$x\,dx = dt$

$z e^{x^2/2} = 2\{t e^t - e^t\} + c$

$\tan y = 2\left[\frac{x^2}{2} - 1\right] + c e^{-x^2/2}$

$\boxed{\tan y = (x^2 - 2) + c e^{-x^2/2}}$

Q $e^y\left[\frac{dy}{dx} + 1\right] = e^x$

Sol $e^y \frac{dy}{dx} + e^y = e^x$

let $e^y = z \Rightarrow e^y \frac{dy}{dx} = \frac{dz}{dx}$

$\frac{dz}{dx} + z = e^x$

$IF = e^{\int 1\,dx} = e^x$

$z \cdot e^x = \int e^x \cdot e^x dx + c$

$$z e^{x} = \frac{e^{2x}}{2} + c$$

$$\boxed{e^{y} = \frac{e^{x}}{2} + c e^{-x}}$$

Equation solvable for P

$\frac{dy}{dx} = P$ first of all factorise the given eqn.

Q① $y\left(\frac{dy}{dx}\right)^2 + (x-y)\frac{dy}{dx} - x = 0$

$$yp^2 + xp - yp - x = 0$$

$$p[yp+x) - 1[yp+x) = 0$$

$$(p-1)(yp+x) = 0$$

$$p = 1, \quad yp + x = 0 \Rightarrow p = -\frac{x}{y}$$

$$\frac{dy}{dx} = 1$$

$$\int dy = \int dx + c$$

$$\boxed{y = x + c}$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\int y\,dy + \int x\,dx = 0$$

$$\frac{y^2}{2} + \frac{x^2}{2} = c_1$$

$$\boxed{y^2 + x^2 = c_2}$$

$$\boxed{x^2 + y^2 = c_2}$$

Q $y = x[p + \sqrt{1+p^2}]$.

$$y - xp = x\sqrt{1+p^2}.$$

square both side

$$y^2 + \cancel{x^2p^2} - 2xyp = x^2 + \cancel{x^2p^2}$$

$$p = \frac{y^2 - x^2}{2xy}$$

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$$\frac{dy}{dx} = \frac{y}{2x} - \frac{x}{2y}$$

or $$\frac{dy}{dx} - \frac{y}{2x} = -\frac{x}{2y}$$

$$2y\frac{dy}{dx} - \frac{y^2}{x} = -x$$

let $y^2 = z \Rightarrow 2y\frac{dy}{dx} = \frac{dz}{dx}$

$$\frac{dz}{dx} - \frac{1}{x}z = -x$$

$$If = e^{\int -\frac{1}{x}dx} = e^{-\log x} = \frac{1}{x}$$

$$z\, If = \int Q \cdot If\, dx + c.$$

$$z\frac{1}{x} = \int -\cancel{x}\frac{1}{\cancel{x}}\,dx + c$$

Putting the value of z.

$$\frac{y^2}{x} = -x + c$$

$$y^2 = -x^2 + cx \rightarrow \boxed{y^2 = cx - x^2}$$

Q $$xy\left(\frac{dy}{dx}\right)^2 - (x^2+y^2)\frac{dy}{dx} + xy = 0$$

Sol Sol $$xy\,p^2 - (x^2+y^2)p + xy = 0$$

$$p = \frac{+(x^2+y^2) \pm \sqrt{(x^2+y^2)^2 - 4x^2y^2}}{2xy}$$

$$p = \frac{x^2+y^2 \pm x^2-y^2}{2xy}$$

$$p = \frac{x^2+y^2+x^2-y^2}{2xy} = \frac{\cancel{2}x^{\cancel{2}}}{\cancel{2}\cancel{x}y} = \frac{x}{y}$$

$\frac{dy}{dx} = \frac{x}{y} \Rightarrow \int y\,dy = \int x\,dx \Rightarrow \boxed{y^2 - x^2 = c}$

again

$P = \frac{x^2+y^2-x^2+y^2}{2yx} = \frac{y}{x}$

$\frac{dy}{dx} = \frac{y}{x} \Rightarrow \int \frac{dy}{y} = \int \frac{dx}{x} \Rightarrow \log x = \log y.$

$\boxed{x/y = c}$

§ Equation solvable for y

If we solve the eq^n for y.

differentiate give eq^n w.r.to x. & putting

$\frac{dy}{dx} = p$ eq^n can convert into P, x

& solve this

Q $y = x + a\tan^{-1} P.$ diff w.r.to x

$\frac{dy}{dx} = 1 + a\frac{1}{1+P^2}\frac{dP}{dx}$

$P - 1 = \frac{a}{1+P^2}\frac{dP}{dx}$

$dx = \frac{a}{(P-1)(P^2+1)}\,dP$ —(1)

Now $\frac{1}{(P-1)(P^2+1)} = \frac{A}{P-1} + \frac{BP+C}{P^2+1}$

$1 = (P^2+1)A + (P-1)(BP+C)$

$A = \frac{1}{2}, \quad B = -\frac{1}{2}, \quad C = \frac{1}{2}$

$\frac{dy}{dx} = \frac{x}{y} \Rightarrow \int y\,dy = \int x\,dx \Rightarrow \boxed{y^2 - x^2 = c}$

again

$$P = \frac{x^2+y^2-x^2+y^2}{2yx} = \frac{y}{x}$$

$\frac{dy}{dx} = \frac{y}{x} \Rightarrow \int \frac{dy}{y} = \int \frac{dx}{x} \Rightarrow \log x = \log y + \log c$

$\boxed{x/y = c}$

Equation solvable for y

If we solve the eqⁿ for y.
differentiate give eqn w.r.to x. & putting $\frac{dy}{dx} = p$ eqⁿ can convert in to P, x
so solve this

Q $y = x + a\tan^{-1}P$. diff w.r.to x

$$\frac{dy}{dx} = 1 + a\frac{1}{1+P^2}\frac{dP}{dx}$$

$$P - 1 = \frac{a}{1+P^2}\frac{dP}{dx}$$

$$dx = \frac{a}{(P-1)(P^2+1)}\,dP \quad \text{—(1)}$$

Now $\frac{1}{(P-1)(P^2+1)} = \frac{A}{P-1} + \frac{BP+C}{P^2+1}$

$$1 = (P^2+1)A + (P-1)(BP+C)$$

$A = \frac{1}{2}$, $B = -\frac{1}{2}$, $C = \frac{1}{2}$

hen $\int dx = \frac{a}{2}\left[\int\left(\frac{1}{P-1} - \frac{P}{P^2+1} + \frac{1}{P^2+1}\right)\right]dP$ (11)

$$x+c = \frac{a}{2}\left[\log(P-1) - \frac{1}{2}\log(P^2+1) + \tan^{-1}P\right]$$

$$\boxed{x+c = \frac{a}{2}\left[\log\frac{P-1}{\sqrt{P^2+1}} + \tan^{-1}P\right]}$$

xP $P=?$ $P=xc$

Q $xP^2+x = 2yP$ — (1)

Sol $2y = xP + \frac{x}{P}$.

diff wrto x

$$2\frac{dy}{dx} = P + x\frac{dP}{dx} - \frac{x}{P^2}\frac{dP}{dx} + \frac{1}{P}$$

$$2P = P + \frac{1}{P} + x\left(1-\frac{1}{P^2}\right)\frac{dP}{dx}$$

$$P - \frac{1}{P} = x\left(1-\frac{1}{P^2}\right)\frac{dP}{dx}$$

$$\frac{\cancel{P^2-1}}{\cancel{P}} = x\frac{(\cancel{P^2-1})}{P^{\cancel{2}}}\frac{dP}{dx}$$

$$\int\frac{dx}{x} = \int\frac{dP}{P}$$

$$\log x + \log c = P \Rightarrow P = xc$$

Putting in (1)

$$x\,x^2c^2 + x = 2ycx$$

$$\boxed{x^2c^2+1 = 2yc}$$

Q $x^2\left(\frac{dy}{dx}\right)^4 + 2x\frac{dy}{dx} - y = 0$ — (1)

Sol or $y = x^2p^4 + 2xp$

diff. w.r.to x

$$\frac{dy}{dx} = 2xp^4 + 4x^2p^3\frac{dp}{dx} + 2p + 2x\frac{dp}{dx}$$

$$p + 2xp^4 + 2x\frac{dp}{dx} + 4x^2p^3\frac{dp}{dx} = 0$$

$$p(1 + 2xp^3) + 2x\frac{dp}{dx}(1 + 2xp^3) = 0$$

$$\left(p + 2x\frac{dp}{dx}\right)(1 + 2xp^3) = 0$$

This give $p + 2x\frac{dp}{dx} = 0$

$$\int\frac{dx}{x} + \int 2\frac{dp}{p} = \text{Const}$$

$$\log x + 2\log p = \log c$$

$xp^2 = c \Rightarrow p = \sqrt{c/x}$ Putting (1)

$$\cancel{x^2}\frac{c^2}{\cancel{x^2}} + 2x\sqrt{c/x} - y = 0$$

$$\boxed{c^2 + 2\sqrt{xc} - y = 0}$$

$$y\sqrt{P^2-1} = -2a\int\sqrt{(P^2-1)}\,dP - \int\frac{2a}{\sqrt{P^2-1}}\,dP$$

$$y\sqrt{P^2-1} = -\frac{\cancel{2}a P\sqrt{P^2-1}}{\cancel{2}} + \frac{\cancel{2}a}{\cancel{2}}\cosh^{-1}x - 2a\cosh^{-1}x$$

$$y\sqrt{P^2-1} = -aP\sqrt{P^2-1} - \cancel{2}a\cosh^{-1}x$$

$$\boxed{y + aP\sqrt{P^2-1} + a\cosh^{-1}x = C}$$ Ans

Q $p^3y + 2px = y$

Sol $2x = \frac{y-p^3y}{\cancel{p}}$

diff wrt y

$$2\frac{dx}{dy} = \frac{p\left[1-p^3-3p^2y\frac{dp}{dy}\right]-(y-p^3y)\frac{dp}{dy}}{p^2}$$

$$\frac{2}{p} = \frac{p-p^4-3p^3y\frac{dp}{dy} - (y+p^3y)\frac{dp}{dy}}{p^{\cancel{2}}}$$

$$P+P^4 = -y\frac{dp}{dy}\left[3p^3+1-p^3\right]$$

$$P+P^4 = -y(1+2p^3)\frac{dp}{dy}$$

$$\int\frac{dy}{-y} = \int\left(\frac{1+2p^3}{P+P^4}\right)dp$$

complete it.

eqⁿ Solvable for x

Given eqⁿ differentiate w.r.t. y and putting $\frac{dx}{dy} = \frac{1}{p}$ & solve eqⁿ for P.

Q $x - yp = ap^2$

Sol $x = yp + ap^2$

diff w.r.t. y

$$\frac{dx}{dy} = y\frac{dp}{dy} + p + 2ap\frac{dp}{dy}$$

$$-p + \frac{1}{p} = (y + 2ap)\frac{dp}{dy}$$

$$\frac{1-p^2}{p}\frac{dy}{dp} = y + 2ap$$

$$\frac{(1-p^2)}{p}\frac{dy}{dp} - y = 2ap$$

$$\frac{dy}{dp} + \frac{p}{p^2-1}y = -\frac{2ap^2}{p^2-1}$$

$$If = e^{\int \frac{p}{p^2-1}dp} = e^{\frac{1}{2}\log(p^2-1)} = \sqrt{p^2-1}$$

$$y \cdot \sqrt{p^2-1} = \int \frac{-2p^2a + 2a - 2a}{\sqrt{p^2-1}}dp + c$$

$$= \int \frac{-2a(p^2-1)}{\sqrt{p^2-1}}dp - \int \frac{2a}{\sqrt{p^2-1}}dp + c$$

Putting, $y\sqrt{p^2-1} = -2a\int \sqrt{p^2-1}\,dp - 2a\int \frac{1}{\sqrt{p^2-1}}\,dp + c$

Now Use the formulla

$$\left[\int \sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a}\right]$$

$$\left[\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a}\right]$$

So $y\sqrt{p^2-1} = -2a\left[\frac{p\sqrt{p^2-1}}{2} - \frac{1}{2}\cosh^{-1}p\right] - 2a\cosh^{-1}p + c$

$$y\sqrt{p^2-1} = -ap\sqrt{p^2-1} + a\cosh^{-1}p - 2a\cosh^{-1}p + c$$

$$(y+ap)\sqrt{p^2+1} = -a\cosh^{-1}p + c$$

or $(y+ap)\sqrt{p^2+1} + a\cosh^{-1}p = c$